Midterm study guide

Here is an outline of things we've learned so far.

Groups: Foundations

- Definition of a group
- Basic properties that follow from the definition (e.g. identity and inverses are unique, $ab = ac \Rightarrow b = c$, etc.)
- order of a group, order of an element.

Dihedral groups

- Definition of D_{2n}
- properties of D_{2n}:
  - What are the elements and their orders?
  - set of generators?
  - which elements commute w/ each other
  - relations? i.e. $sr^i = r^{-i}s$

Symmetric groups

- A a set, definition of S_A (elements are bijections)
- Definition of S_n

- cycle decompositions in S_n, and how to multiply elts given their cycle decomps. (e.g. What is $(1\,2\,3)(2\,4\,3\,1)$?)

Homomorphisms

- Definition of homomorphism, isomorphism.

- How to check that a homomorphism is an iso?

  Show it's bijective or if G and H are both finite of same order, show $\varphi: G \to H$ is either surjective or injective

  (trivial kernel!)

- How to show two groups G, H are not isomorphic? Show that there's some property that one has but not the other.

  e.g.:
  - $|G| \neq |H|$
  - the elements have different orders
  - one's abelian, one's not
  - the centers have different orders
  - etc. (can you come up w/ more ways?)

- If $\varphi: G \to H$ is a homomorphism, then if we know where φ sends the generators of G, we know where it sends every element of G

- Defs of $\ker\varphi$, $\operatorname{im}\varphi$; they're both subgroups

  (exercise: if $X \subseteq G$ is a generating set for G, then

$\varphi(X) = \{\varphi(x) \mid x \in X\}$ generates $\operatorname{im}\varphi$.)

- Definition of automorphism. If G is a group, $\operatorname{Aut}(G)$ is a group

Group actions

- Definition. What is an action of G on a set A?
- If G acts on A, and $g \in G$, then $\sigma_g \in S_A$, defined $\sigma_g(a) = g \cdot a$.
- The map $G \to S_A$ defined $g \longmapsto \sigma_g$ is a homomorphism, called the permutation representation of the action
- kernel of an action; action is faithful if $\ker = 1$. What does a faithful action look like?
- orbits: What are the orbits of an action?
- $G_a :=$ stabilizer of a in G = $g \in G$ s.t. $g \cdot a = a$. $G_a \leq G$.
- G acts on itself by $g \cdot a = ga$; it's faithful
- G acts on itself by conjugation: $g \cdot a = gag^{-1}$.
- G acts on $P(G)$ by conjugation

Subgroups

- subgroup criterion, checking that $H \subseteq G$ is a subgroup

- cyclic subgroups: $\langle x \rangle$. Order of $\langle x \rangle = |x|$.

- subgroup generated by a subset. i.e. if $X \subseteq G$, definition of $\langle X \rangle \leq G$

- special subgroups: $Z(G)$, $C_G(S)$, $N_G(S)$ (for $S \subseteq G$)
- Lagrange's Theorem (G finite, $H \leq G \Rightarrow |H| \mid |G|$)

Cyclic groups

- Definition and properties

- finite vs. infinite cyclic group

  finite: iso to $\mathbb{Z}/n\mathbb{Z}$; infinite: iso to $\mathbb{Z}$

- subgroups of cyclic groups are cyclic, and in $\mathbb{Z}_n$, there's exactly one subgroup for each divisor of n.